# Assembling Quantum Systems

## Architecture

Lecture 5

# Assembling Quantum States

Assume we have two independent quantum systems *Q* and *Q'*, represented respectively by the vector spaces ***V*** and ***V'***. The quantum system obtained by merging *Q* and *Q'* will have the tensor product ***V*** and ***V'*** as a state space.

We can assemble as many systems as we like: $V_0 \otimes V_1 \otimes \cdots \otimes V_k$

Two particles moving in a one-dimensional grid: first particle on $\{x_0, x_1, \ldots, x_{n-1}\}$, the second particle on $\{y_0, y_1, \ldots, y_{m-1}\}$.


*n* x *m* possible basic states:

$|x_0\rangle \otimes |y_0\rangle$, meaning the first particle is at $x_0$, and the second particle at $y_0$.

⋮

$|x_{n-1}\rangle \otimes |y_{m-1}\rangle$, meaning the first particle is at $x_{n-1}$ and the second at $y_{m-1}$.

# Assembling (cont'd)

- Generic state vector:

  $$|\phi\rangle = c_{0,0}\,|x_0\rangle \otimes |y_0\rangle + \cdots c_{i,j}\,|x_i\rangle \otimes |y_j\rangle + \cdots + c_{n-1,m-1}\,|x_{n-1}\rangle \otimes |y_{m-1}\rangle$$

  which is a vector in the (*n* x *m*)-dimensional complex space $\boldsymbol{C}^{nxm}$.

- The quantum amplitude $|c_{i,j}|$ squared is the probability of finding the two particles at positions $x_i$ and $y_j$.

- Example:

  $$|\phi\rangle = i\,|x_0\rangle \otimes |y_0\rangle + (1-i)\,|x_0\rangle \otimes |y_1\rangle + 2\,|x_1\rangle \otimes |y_0\rangle + (-1-i)\,|x_1\rangle \otimes |y_1\rangle$$

- What is probability of finding first particle at $x_1$ and second one at $y_1$?

  $$p(x_1, y_1) = \frac{|-1-i|^2}{|i|^2 + |1-i|^2 + |2|^2 + |-1-i|^2} = 0.2222$$

# Assembling (cont'd)

- Entanglement:
  - The basic states of the assembled system are just the tensor product of basic states of its constituents.
  - *Each* generic state vector can be rewritten as the tensor product of two states, coming from one subsystem and a second one? **NOT TRUE**

- Example: simplest two-particle system, where each particle is allowed only in two points. Consider the state

  $$|\phi\rangle = |x_0\rangle \otimes |y_0\rangle + |x_1\rangle \otimes |y_1\rangle$$

  In order to clarify what is left out, we might write this as

  $$|\phi\rangle = 1\,|x_0\rangle \otimes |y_0\rangle + 0\,|x_0\rangle \otimes |y_1\rangle + 0\,|x_1\rangle \otimes |y_0\rangle + 1\,|x_1\rangle \otimes |y_1\rangle$$

- Can we write this as the tensor product of two states coming from two subsystems? 1st particle $c_0\,|x_0\rangle + c_1\,|x_1\rangle$, 2nd particle $c_0'\,|y_0\rangle + c_1'\,|y_1\rangle$
  Tensor product

  $$(c_0\,|x_0\rangle + c_1\,|x_1\rangle) \otimes (c_0'\,|y_0\rangle + c_1'\,|y_1\rangle) = c_0c_0'\,|x_0\rangle \otimes |y_0\rangle + c_0c_1'\,|x_0\rangle \otimes |y_1\rangle + c_1c_0'\,|x_1\rangle \otimes |y_0\rangle + c_1c_1'\,|x_1\rangle \otimes |y_1\rangle$$

- No solution: $|\psi>$ cannot be written as a tensor product.

**Entangled states**

# Assembling: Entanglement

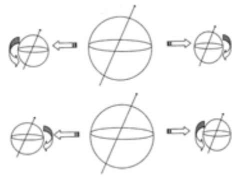
- $|\phi\rangle = |x_0\rangle \otimes |y_0\rangle + |x_1\rangle \otimes |y_1\rangle$   What does it physically mean?
- First particle 50-50 chance of being in $x_0$ or $x_1$.
- If in $x_0$? Term $|x_0\rangle \otimes |y_1\rangle$ has coefficient 0, so no chance that second particle in $y_1$. We must conclude that it can only be found in $y_0$.
- Similarly, if first particle in $x_1$, second one must be in $y_1$.
- Symmetrical with respect to the two particles: the same if we measure second particle first.
- The individual states of the two particles are intimately related to each other, or entangled.
- Amazing: the $x_i$'s can be *light years* away from the $y_i$'s!
- Sharp contrast: no clue

  **Separable states**

  $$|\phi'\rangle = 1\,|x_0\rangle \otimes |y_0\rangle + 1\,|x_0\rangle \otimes |y_1\rangle + 1\,|x_1\rangle \otimes |y_0\rangle + 1\,|x_1\rangle \otimes |y_1\rangle$$

# Assembling: spin systems

- Law of conservation of spin: in an isolated system the total amount of spin must stay the same.
- Fix on the *z*-direction and corresponding spin basis: up and down.
- Consider a composite particle, whose total spin is zero.
- This particle might split up into two particles that do have spin:



# Assembling: spin systems (cont'd)

- Spin states of the two particles will be entangled.
- Spin of total system zero → sum of the spins of the two particles must cancel each other out:
  - Measure spin of left particle along *z* axis $|\uparrow_L\rangle$ → spin of right particle $|\downarrow_R\rangle$
  - Similarly, $|\downarrow_L\rangle \rightarrow |\uparrow_R\rangle$
- Basis left particle $B_L=\{|\uparrow_L\rangle, |\downarrow_L\rangle\}$, basis right particle $B_R=\{|\uparrow_R\rangle, |\downarrow_R\rangle\}$, so basis of total system

$$\{\uparrow_L \otimes \uparrow_R,\ \uparrow_L \otimes \downarrow_R,\ \downarrow_L \otimes \uparrow_R,\ \downarrow_L \otimes \downarrow_R\}$$

- Entangled particles are described by $\frac{|\uparrow_L \otimes \downarrow_R\rangle + |\downarrow_L \otimes \uparrow_R\rangle}{\sqrt{2}}$
- Combinations $\uparrow_L \otimes \uparrow_R$ and $\downarrow_L \otimes \downarrow_R$ cannot occur because of the law of conservation of spin.
- Measuring left particle: if it collapses to $|\uparrow_L\rangle$ then instantaneously right particle collapses to $|\downarrow_R\rangle$, even if the particle is millions of light years away.
- Entanglement plays role in: algorithms, cryptography, teleportation, and decoherence.

# Assembling systems

Summarizing:

– We can use the tensor product to build complex systems out of simpler ones.

– The new system cannot be analyzed simply in terms of states belonging to its subsystems. An entire set of new states has been created, which cannot be resolved into their constituents.

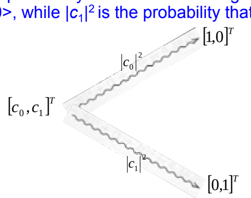
# **Architecture**

# Bits and qubits

- A bit is a unit of information describing a two-dimensional classical system.
- A bit is away of describing a system whose set of states is of size 2, usually written as 0 and 1, or F and T, etc.
- By matrices: $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ and $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$
- In a classical world: either in state |0> or in state |1>; in a quantum world this is not sufficient: a quantum system can be in state |0> and in state |1> simultaneously.
- A quantum bit or a qubit is a unit of information describing a two-dimensional quantum system.

# Qubits: representation

- Representation of a qubit $\begin{bmatrix}c_0\\c_1\end{bmatrix}$ where $|c_0|^2+|c_1|^2=1$
- $|c_0|^2$ is the probability that after measuring the qubit, it will be found in state |0>, while $|c_1|^2$ is the probability that it will be in state |1>




- Canonical basis of $\mathbf{C}^2$: $\begin{bmatrix}c_0\\c_1\end{bmatrix} = c_0 \cdot \begin{bmatrix}1\\0\end{bmatrix} + c_1 \cdot \begin{bmatrix}0\\1\end{bmatrix} = c_0\,|0\rangle + c_1\,|1\rangle$

# Qubits: denotations and implementations

Several ways of denoting qubits

$$\tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}\end{bmatrix} = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}} = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{|1\rangle + |0\rangle}{\sqrt{2}} \qquad \frac{|0\rangle - |1\rangle}{\sqrt{2}} \neq \frac{|1\rangle - |0\rangle}{\sqrt{2}} \qquad \frac{|0\rangle - |1\rangle}{\sqrt{2}} = (-1)\frac{|1\rangle - |0\rangle}{\sqrt{2}}$$

Examples of qubit implementations (see chapter 11)

- An electron in an atom might be in one of two different energy levels (ground state and excited states).
- A photon might be in one of two polarized states.
- A subatomic particle might have one of two spin directions.

There will be enough quantum indeterminacy and quantum superposition effects within all these systems to represent a qubit.

## Qubits: more than 1 bit

- Only one bit of storage not very interesting. Consider a byte or eight bits 01101011
- Following the preceding method of describing bits

$$\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}$$

- To combine quantum systems one should use tensor products

$$|0\rangle\otimes|1\rangle\otimes|1\rangle\otimes|0\rangle\otimes|1\rangle\otimes|0\rangle\otimes|0\rangle\otimes|1\rangle$$

- This is an element of

$$\mathbf{C}^2\otimes\mathbf{C}^2\otimes\mathbf{C}^2\otimes\mathbf{C}^2\otimes\mathbf{C}^2\otimes\mathbf{C}^2\otimes\mathbf{C}^2\otimes\mathbf{C}^2$$

- This vector space may be denoted as $(\mathbf{C}^2)^{\otimes 8}$. This is a complex vector space of dimension $2^8$=256, isomorphic to $\mathbf{C}^{256}$.

## Qubyte

- Another description: a $2^8$=256 row vector

$$\begin{matrix}00000000\\00000001\\\vdots\\01101010\\01101011\\01101100\\\vdots\\11111110\\11111111\end{matrix}\begin{bmatrix}0\\0\\\vdots\\0\\1\\0\\\vdots\\0\\0\end{bmatrix}\qquad\begin{matrix}00000000\\00000001\\\vdots\\01101010\\01101011\\01101100\\\vdots\\11111110\\11111111\end{matrix}\begin{bmatrix}c_0\\c_1\\\vdots\\c_{106}\\c_{107}\\c_{108}\\\vdots\\c_{254}\\c_{255}\end{bmatrix}\qquad\sum_{i=0}^{255}|c_i|^2=1$$

- Classical world: indicate the state of each bit of a byte → eight bits.
- Quantum world: a state of eight qubits is given by writing 256 complex numbers.
- A 64-qubit register: $2^{64}$ = 18,446,744,073,709,551,616 complex numbers.
- Exponential growth: thought to the notion of quantum computing.

## Two-qubit system

- Qubit pair: $|0\rangle\otimes|1\rangle$ or $|0\otimes 1\rangle$
- Tensor product clear: $|0\rangle|1\rangle, |0,1\rangle$ or $|01\rangle$
- Another way: $\begin{matrix}00\\01\\10\\11\end{matrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$
- A general state of a two-qubit system:

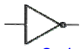
$$|\phi\rangle = c_{0,0}|00\rangle + c_{0,1}|01\rangle + c_{1,0}|10\rangle + c_{1,1}|11\rangle$$

- Tensor product of two states not commutative:

$$|0\otimes 1\rangle = |0\rangle\otimes|1\rangle = |0,1\rangle = |01\rangle \neq |10\rangle = |1,0\rangle = |1\rangle\otimes|0\rangle = |1\otimes 0\rangle$$

- Entangled states: $\dfrac{|11\rangle+|00\rangle}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}|11\rangle + \dfrac{1}{\sqrt{2}}|00\rangle$

## Classical gates: NOT

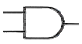
NOT gate 

- Input 1 bit or a 2x1 matrix
- Output 1 bit or a 2x1 matrix
- NOT of $|0\rangle$ equals $|1\rangle$ and NOT of $|1\rangle$ equals $|0\rangle$
- Consider the matrix

$$\text{NOT} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\qquad\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

## Classical gates: AND

AND gate 

- Input 2 bits, output 1 bit, so a $2^1$-by-$2^2$ matrix
- Consider the matrix

$$\text{AND} = \begin{bmatrix}1 & 1 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\quad\text{or}\quad \text{AND}|11\rangle = |1\rangle$$

$$\begin{bmatrix}1 & 1 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}3.5\\2\\0\\-4.1\end{bmatrix}=\begin{bmatrix}5.5\\-4.1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\quad\text{or}\quad \text{AND}|01\rangle = |0\rangle$$

NONSENS! Only classical states, i.e. columns matrices with a single 1 entry and all other entries 0. Later more…..

## Classical gates: OR

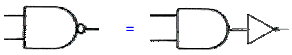
OR gate 

- Consider the matrix

$$\text{OR} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 1 & 1\end{bmatrix}$$

 ?

## Classical gates: NAND

NAND gate 

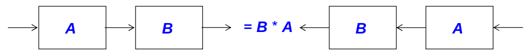
- Special importance because every logical gate can be composed of NAND gates.

$$\text{NAND} = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

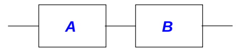
- NAND = AND followed by NOT

$$\text{NOT} * \text{AND} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \text{NAND}$$

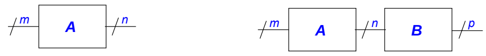
## Sequential operations




Convention: computation flows from left to right,
so A followed by B shall be denoted as

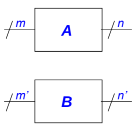



*m* input bits and *n* output bits



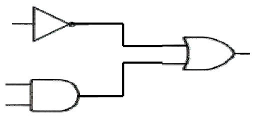

***A*** will be of size $2^n$-by-$2^m$ ***B*A*** is a ($2^p$-by-$2^n$)*($2^n$-by-$2^m$)= ($2^p$-by-$2^m$) matrix

## Parallel operations

$\mathbf{A} \otimes \mathbf{B} =$ 


$\mathbf{A} \otimes \mathbf{B}$ is of size $2^n 2^{n'} = 2^{n+n'}$ - by - $2^m 2^{m'} = 2^{m+m'}$

## Parallel operations: example

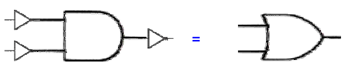


$\text{OR} * (\text{NOT} \otimes \text{AND})$

$$\text{NOT} \otimes \text{AND} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{OR} * (\text{NOT} \otimes \text{AND}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

## Example DeMorgan's laws

$\neg(\neg P \wedge \neg Q) = P \vee Q$ 

$\text{NOT} * \text{AND} * (\text{NOT} \otimes \text{NOT}) = \text{OR}$

$$\text{NOT} \otimes \text{NOT} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

## Reading

- This lecture: Ch 4.5 & Ch 5.1-5.2, p 132-151.
- Next lecture: Ch 5.3-5.5