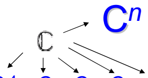
Complex Vector Spaces

Lecture 2

C^n as an example



- $C^4 = C \times C \times C \times C$, the vectors of length 4
- E.g. $$V = \begin{bmatrix} 6-4i \\ 7+3i \\ 4.2-8.1i \\ -3i \end{bmatrix}, \text{where } V[1] = 7+3i \text{ (start with 0 as index)}$$
- Addition $(V + W)[j] = V[j] + W[j]$
  - Commutative $V + W = W + V$
  - Associative $(V + W) + X = V + (W + X)$
  - Zero vector $V + \mathbf{0} = \mathbf{0} + V = V$
  - (Additive) inverse or negative $W + (-W) = (-W) + W = \mathbf{0}$

  Set with these properties is called an Abelian group.

Complex vector space

- Complex number c (a scalar)
- Multiplication of a scalar and a vector $(c \cdot V)[j] = c \times V[j]$, where x is the complex multiply
- Properties
  - $1 \cdot V = V$
  - $c_1 \cdot (c_2 \cdot V) = (c_1 \times c_2) \cdot V$
  - $c \cdot (V + W) = c \cdot V + c \cdot W$
  - $(c_1 + c_2) \cdot V = c_1 \cdot V + c_2 \cdot V$

  An Abelian group with these properties is called a complex vector space.

Formal definition

A complex vector space is a nonempty set **V**, whose elements we call vectors, with three operations

- Addition: $+ : \mathbf{V} \times \mathbf{V} \rightarrow \mathbf{V}$
- Negation: $- : \mathbf{V} \rightarrow \mathbf{V}$
- Scalar multiplication: $\cdot : \mathbf{C} \times \mathbf{V} \rightarrow \mathbf{V}$

and a distinguished element called the zero vector **0**.

They must satisfy the following properties:

i. Commutativity of addition: $V + W = W + V$
ii. Associativity of addition: $(V + W) + X = V + (W + X)$
iii. Zero is an additive identity: $V + \mathbf{0} = V = \mathbf{0} + V$
iv. Every vector has an inverse: $V + (-V) = \mathbf{0} = (-V) + V$
v. Scalar multiplication has a unit: $1 \cdot V = V$
vi. Scalar multiplication respects complex multiplication: $c_1 \cdot (c_2 \cdot V) = (c_1 \times c_2) \cdot V$
vii. Scalar multiplication distributes over addition: $c \cdot (V + W) = c \cdot V + c \cdot W$
viii. Scalar multiplication distributes over complex addition: $(c_1 + c_2) \cdot V = c_1 \cdot V + c_2 \cdot V$

Properties i, ii, iii, and iv: Abelian group;
all properties: complex vector space.

Real vector space

A real vector space is a nonempty set V, analogue to a complex vector space, but there is a scalar multiplication that uses R and not C, i.e.,

$\cdot : R \times V \rightarrow V$.

This set and these operations must satisfy the analogous properties of a complex vector space.

C^n

- C^n, the set of vectors of length n with complex entries, will be complex vector space that serves as primary example for the class.
- It is also a real vector space, because every complex vector space is also a real vector space.
- R^n, the set of vectors of length n with real entries, is a real vector space.

$C^{m \times n}$

- $C^{m \times n}$, the set of all m-by-n matrices with complex entries, is a complex vector space.
- $A \in C^{m \times n}$

$$A = \begin{bmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,n-1} \\ c_{1,0} & c_{1,1} & \cdots & c_{0,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m-1,0} & c_{m-1,1} & \cdots & c_{m-1,n-1} \end{bmatrix}$$

- Addition: $(A + B)[j,k] = A[j,k] + B[j,k]$
- Inverse: $(-A)[j,k] = -(A[j,k])$
- Scalar multiplication: $(c \cdot A)[j,k] = c \times A[j,k]$

Operations on matrices

- The transpose of A, denoted A^T, is defined as

  $A^T[j,k] = A[k,j]$

- The conjugate of A, denoted $\overline{A}$, is defined as

  $\overline{A}[j,k] = \overline{A[j,k]}$

- Combined gives this the adjoint or dagger operation $A^\dagger$, defined as

  $A^\dagger = (\overline{A})^T = \overline{(A^T)}$ or $A^\dagger[j,k] = \overline{A[k,j]}$

Properties

- Transpose is idempotent: $(A^T)^T = A$
- Transpose respects addtion: $(A + B)^T = A^T + B^T$
- Transpose respects scalar multiplication: $(c \cdot A)^T = c \cdot A^T$

- Conjugate is idempotent: $\overline{\overline{A}} = A$
- Conjugate respects addtion: $\overline{A + B} = \overline{A} + \overline{B}$
- Conjugate respects scalar multiplication: $\overline{c \cdot A} = \overline{c} \cdot \overline{A}$

- Adjoint is idempotent: $(A^\dagger)^\dagger = A$
- Adjoint respects addtion: $(A + B)^\dagger = A^\dagger + B^\dagger$
- Adjoint respects scalar multiplication: $(c \cdot A)^\dagger = \overline{c} \cdot A^\dagger$

Matrix multiplication

- Matrix multiplication is a binary operation

  $* : C^{m \times n} \times C^{n \times p} \rightarrow C^{m \times p}$

- Formally

$$(A * B)[j,k] = \sum_{h=0}^{n-1} (A[j,h] \times B[h,k])$$

- When it is clear * will be omitted.

Properties of matrix multiplication

- Associative: $(A * B) * C = A * (B * C)$
- I_n as unit: $I_n * A = A = A * I_n$ with I_n identity matrix
- Distributes over addition:
  $A * (B + C) = (A * B) + (A * C)$
  $(B + C) * A = (B * A) + (C * A)$
- Respects scalar multiplication:
  $c \cdot (A * B) = (c \cdot A) * B = A * (c \cdot B)$
- Relates to the transpose: $(A * B)^T = B^T * A^T$
- Respects the conjugate: $\overline{A * B} = \overline{A} * \overline{B}$
- Relates to the adjoint: $(A * B)^\dagger = B^\dagger * A^\dagger$
- Note: commutativity is not a basic property!
- A complex vector space V with a multiplication * that satisfies the first four properties is called a complex algebra.

Complex subspace

- Given two complex vector spaces V and V', we say that V is a complex subspace of V' if V is a subset of V and the operations of V are restrictions of operations of V'.

Linear map/operator/isomorphism

- Let V and V' be two complex vector spaces. A linear map from V to V' is a function f : V → V' such that
  - f respects addition: $f(V_1+V_2) = f(V_1) + f(V_2)$
  - f respects the scalar multiplication: $f(c \cdot V) = c \cdot f(V)$
- A linear map from a complex vector space to itself is called an operator. If $F(V) = A^*V$ is an operator, we say that F is represented by A.
- Two complex vector spaces V and V' are isomorphic if there is a one-to-one linear map f : V → V'. Such a map is called an isomorphism. When two vector spaces are isomorphic, it means that the names of the elements of the vector spaces are renamed but the structure of the two spaces are the same. Two such vector spaces are "essentially the same".

Basis

- Linear combination:
  $V = c_0 \cdot V_0 + c_1 \cdot V_1 + \cdots + c_{n-1} \cdot V_{n-1}$.
- Linearly independent if
  $\mathbf{0} = c_0 \cdot V_0 + c_1 \cdot V_1 + \cdots + c_{n-1} \cdot V_{n-1}$
  implies that $c_0 = c_1 = \cdots = c_{n-1} = 0$.

  Is equivalent that for any nonzero V there are unique coefficients $c_0, c_1, \cdots, c_{n-1}$ such that
  $V = c_0 \cdot V_0 + c_1 \cdot V_1 + \cdots + c_{n-1} \cdot V_{n-1}$.
- A set B = $\{V_0, V_1, \ldots, V_{n-1}\}$ of vectors is called a basis of a (complex) vector space V if both
  i. Every V can be written as a linear combination of vectors from B
  ii. B is linearly independent.

Canonical or standard basis

- R^3: $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$
- C^n (and R^n): $E_0 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix},\ E_1 = \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \ldots,\ E_{n-2} = \begin{bmatrix}0\\\vdots\\1\\0\end{bmatrix},\ E_{n-1} = \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}.$
- See book for matrices and others.

Dimension

- Bases for R^3, e.g.:

$$\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\} \quad \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\} \quad \left\{ \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}3\\-2\\0\end{bmatrix} \right\}$$

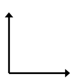
- For every vector space, every basis has the same number of vectors. This is called the dimension of the vector space.

Transition matrix

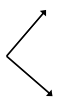
- A change of basis matrix or a transition matrix from basis B to basis D is a matrix $M_{D \leftarrow B}$ such that for any vector V we have

  $V_D = M_{D \leftarrow B} * V_B$

- Important example: Hadamard matrix



$$\Leftarrow \quad H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix} \quad \Rightarrow$$



Inner product

Inner product (or dot product or scalar product) on a complex vector space V is a function

$$\langle -,- \rangle : \mathbf{V} \times \mathbf{V} \to \mathbf{C}$$

satisfying the following conditions

i. Nondegenerate $\langle \mathbf{V}, \mathbf{V} \rangle \geq 0,$
$\langle \mathbf{V}, \mathbf{V} \rangle = \mathbf{0}$ if and only if $\mathbf{V} = \mathbf{0}$

ii. Respects addition $\langle \mathbf{V}_1 + \mathbf{V}_2, \mathbf{V}_3 \rangle = \langle \mathbf{V}_1, \mathbf{V}_3 \rangle + \langle \mathbf{V}_2, \mathbf{V}_3 \rangle,$
$\langle \mathbf{V}_1, \mathbf{V}_2 + \mathbf{V}_3 \rangle = \langle \mathbf{V}_1, \mathbf{V}_2 \rangle + \langle \mathbf{V}_1, \mathbf{V}_3 \rangle.$

iii. Respects scalar multiplication $\langle c \cdot \mathbf{V}_1, \mathbf{V}_2 \rangle = \bar{c} \times \langle \mathbf{V}_1, \mathbf{V}_2 \rangle,$
$\langle \mathbf{V}_1, c \cdot \mathbf{V}_2 \rangle = c \times \langle \mathbf{V}_1, \mathbf{V}_2 \rangle.$

iv. Skew symmetric
$\langle \mathbf{V}_1, \mathbf{V}_2 \rangle = \overline{\langle \mathbf{V}_2, \mathbf{V}_1 \rangle}$

Examples

$\mathbf{R}^n : \langle \mathbf{V}_1, \mathbf{V}_2 \rangle = \mathbf{V}_1^T * \mathbf{V}_2$

$\mathbf{C}^n : \langle \mathbf{V}_1, \mathbf{V}_2 \rangle = \mathbf{V}_1^\dagger * \mathbf{V}_2$

$\mathbf{C}^{n \times n} : \langle \mathbf{A}, \mathbf{B} \rangle = \mathrm{Trace}(\mathbf{A}^\dagger * \mathbf{B})$, where $\mathrm{Trace}(\mathbf{C}) = \sum_{i=0}^{n-1} \mathbf{C}[i,i]$

See book for other examples

Norm or length

Norm or length is a function $|\ | : \mathbf{V} \to \mathbf{R}$
defined as $|\mathbf{V}| = \sqrt{\langle \mathbf{V}, \mathbf{V} \rangle}$

i. Norm is nondegenerate: $|\mathbf{V}| > 0$ if $\mathbf{V} \neq \mathbf{0}$ and $|\mathbf{0}| = 0$
ii. Norm satisfies the triangle inequality: $|\mathbf{V} + \mathbf{W}| \leq |\mathbf{V}| + |\mathbf{W}|$
iii. Norm respects scalar multiplication: $|c \cdot \mathbf{V}| = |c| \times |\mathbf{V}|$

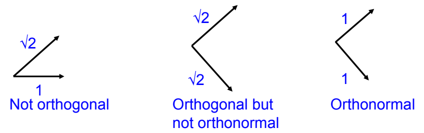
Distance function

Distance function is a function $d(\ ,\ ) : \mathbf{V} \times \mathbf{V} \to \mathbf{R}$
where $d(\mathbf{V}_1, \mathbf{V}_2) = |\mathbf{V}_1 - \mathbf{V}_2| = \sqrt{\langle \mathbf{V}_1 - \mathbf{V}_2, \mathbf{V}_1 - \mathbf{V}_2 \rangle}$

i. Distance is nondegenerate:
$d(\mathbf{V}, \mathbf{W}) > 0$ if $\mathbf{V} \neq \mathbf{W}$ and $d(\mathbf{V}, \mathbf{V}) = 0$
ii. Distance satisfies the triangle inequality:
$d(\mathbf{U}, \mathbf{V}) \leq d(\mathbf{U}, \mathbf{W}) + d(\mathbf{W}, \mathbf{V})$
iii. Distance is symmetric:
$d(\mathbf{V}, \mathbf{W}) = d(\mathbf{W}, \mathbf{V})$

Orthogonal and orthonormal basis

- Orthogonal basis B = $\{V_0, V_1, \ldots, V_{n-1}\}$:
  vectors pairwise orthogonal, $j \neq k$ implies $\langle V_j, V_k \rangle = 0$
- Orthonormal basis B:
  orthogonal and every basis vector is of norm 1



Hilbert space

- A Hilbert space is a complex inner product space that is complete (for definition see book).
- Every finite-dimensional complex vector space with an inner product is automatically a Hilbert space.

Errata chapter 2

All errata:
http://www.cambridge.org/resources/0521879965/7337_Errata.pdf

This link will be available soon on the QC-webpage.

Reading

- This lecture: Ch 2.1-2.4, p 29-60.
- Next lecture: Ch 2.5-2.7 & (start of) Ch 3.