Complex Numbers

Lecture 1

Number systems

- Positive numbers, $P = \{1,2,3,\ldots\}$
- Natural numbers, $N = \{0,1,2,3,\ldots\}$
- Integers (or whole numbers), $Z = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$
- Rational numbers, $Q = \left\{ \frac{m}{n} \middle| m \in Z, n \in P \right\}$
- Real numbers, $R = Q \cup \left\{ \ldots, \sqrt{2}, \ldots, e, \ldots, \pi, \ldots, \frac{e}{\pi}, \ldots \right\}$
- New system: complex numbers

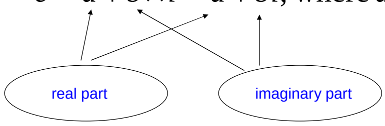
Imaginary numbers

$$x^2 + 1 = 0; x?$$
$$\Downarrow$$
$$x^2 = -1$$
$$\downarrow \textit{postulate}$$
$$i^2 = -1 \quad \text{or} \quad i = \sqrt{-1}$$

$$\text{Imaginary numbers}: a \times i, a \in R$$

Complex numbers

$$c = a + b \times i = a + bi, \text{ where } a, b \in R$$



$$\text{set of complex numbers}: C$$

Algebra of complex numbers

Definition : $c \mapsto (a,b)$ ordered pair of reals

real numbers : $a \mapsto (a,0)$

imaginary numbers : $b \mapsto (0,b)$, e.g. $i \mapsto (0,1)$

Addition : $(a_1,b_1)+(a_2,b_2)=(a_1+a_2,b_1+b_2)$

Multiplication : $(a_1,b_1)\times(a_2,b_2)=(a_1,b_1)(a_2,b_2)=$
$=(a_1a_2-b_1b_2,a_1b_2+a_2b_1)$

Algebra (cont'd)

- Addition and multiplication are **commutative:**
  $c_1 + c_2 = c_2 + c_1$ and $c_1 \times c_2 = c_2 \times c_1$
- They are also **associative**:
  $(c_1 + c_2) + c_3 = c_1 + (c_2 + c_3)$ and $(c_1 \times c_2) \times c_3 = c_1 \times (c_2 \times c_3)$
- Multiplication **distributes** over addition:
  $c_1 \times (c_2 + c_3) = (c_1 \times c_2) + (c_1 \times c_3)$

Algebra (cont'd)

Subtraction : $c_1 - c_2 = (a_1, b_1) - (a_2, b_2) = (a_1 - a_2, b_1 - b_2)$

Division : $(x, y) = \dfrac{(a_1, b_1)}{(a_2, b_2)}$

$\Rightarrow (a_1, b_1) = (x, y) \times (a_2, b_2) = (a_2 x - b_2 y, a_2 y + b_2 x)$

So (1) $a_1 = a_2 x - b_2 y \quad \times a_2 \Rightarrow$ (1') $a_1 a_2 = a_2^{\,2} x - b_2 a_2 y$

(2) $b_1 = a_2 y + b_2 x \quad \times b_2 \Rightarrow$ (2') $b_1 b_2 = a_2 b_2 y + b_2^{\,2} x$

(1')+(2') $a_1 a_2 + b_1 b_2 = (a_2^{\,2} + b_2^{\,2})x \quad \Rightarrow x = \dfrac{a_1 a_2 + b_1 b_2}{a_2^{\,2} + b_2^{\,2}}$

In the way : $(1) \times b_2 + (2) \times -a_2 \quad \Rightarrow y = \dfrac{a_2 b_1 - a_1 b_2}{a_2^{\,2} + b_2^{\,2}}$

Algebra (cont'd)

- Absolute value for real numbers:

$$|a| = +\sqrt{a^2}$$

- Generalization for complex numbers:

$$|c| = |a + bi| = +\sqrt{a^2 + b^2}$$

modulus of a complex number

Algebra (cont'd)

$$|c_1||c_2| = |c_1 c_2|$$

$$|c_1 + c_2| \le |c_1| + |c_2|$$

$$c + (0,0) = (0,0) + c = c \;\Rightarrow\; (0,0) \text{ is additive identity}$$

$$c \times (1,0) = (1,0) \times c = c \;\Rightarrow\; (1,0) \text{ is multiplicative identity}$$

Algebra (cont'd)

- Summarizing, defined a set of numbers *C* with 4 operations and following properties:
  1) Addition is commutative and associative
  2) Multiplication is commutative and associative
  3) Addition has identity: (0,0)
  4) Multiplication has identity: (1,0)
  5) Multiplication distributes with respect to addition
  6) Subtraction (i.e., inverse of addition) is defined everywhere
  7) Division (i.e., inverse of multiplication) is defined everywhere except when the divisor is zero.
- → *C* is a
  - **field**
  - **algebraically complete:** contains all solutions for any of its polynominal equations (*R* is not)

Algebra (cont'd)

- Unary operation 'changing sign':
  1) change the sign of the real part
  2) change the sign of the imaginary part
  3) change both
- 3) is obtained by multiplication with (-1,0)
- What about 2) and 1)?

Algebra (cont'd)

- **Conjugation**

  The conjugate of $c = a + bi$ is $\bar{c} = a - bi$.

- Properties:

  Conjugation respects addition : $\overline{c_1} + \overline{c_2} = \overline{c_1 + c_2}$

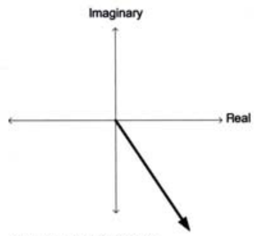
  Conjugation respects multiplication : $\overline{c_1} \times \overline{c_2} = \overline{c_1 \times c_2}$

  Conjugation $c \mapsto \bar{c}$ is bijective

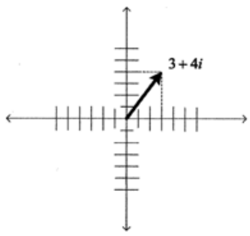
  } field isomorphism

- Changing the sign of the real part has no particular name.

Geometry of complex numbers



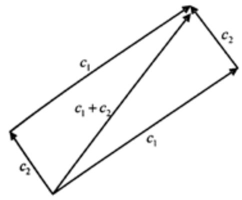
Complex or Argand plane



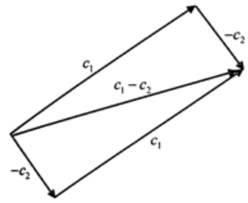
Vector 3 + 4*i*

Modulus is the length of the vector

Geometry (cont'd)



Addition: **Parallelogram rule**

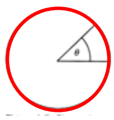


Subtraction

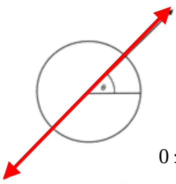
Geometry (cont'd)

- Cartesian representation (*a*,*b*)
- Polar representation (ρ,θ), where ρ represents the modulus/magnitude, and θ is called the angle/phase

$\rho \geq 0$



Points with same ρ



$0 \leq \theta < 2\pi$

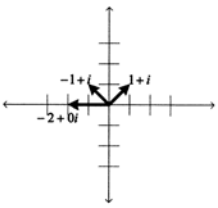
Points with same θ

Geometry (cont'd)

$$\text{From Cartesian} \rightarrow \text{polar} \quad \begin{cases} \rho = \sqrt{(a^2 + b^2)} \\ \theta = \tan^{-1}\left(\frac{b}{a}\right) \end{cases}$$

$$\text{From polar} \rightarrow \text{Cartesian} \quad \begin{cases} a = \rho\cos(\theta) \\ b = \rho\sin(\theta) \end{cases}$$

Geometry (cont'd)



$$\text{Multiplication}: (\rho_1, \theta_1) \times (\rho_2, \theta_2) = (\rho_1\rho_2, \theta_1 + \theta_2)$$

Errata chapter 1

1. Page. xv. Section ACKNOLWEDGMENTS should be ACKNOWLEDGEMENTS. Alex Sverdlov, 10/26/2008.
2. Page 325: Equation (B.3) should be

$$\begin{aligned} |c_1||c_2| &= \sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2} = \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} \\ &= \sqrt{a_1^2 a_2^2 + b_1^2 a_2^2 + b_1^2 b_2^2 + a_1^2 b_2^2} = \sqrt{(\boldsymbol{a_1 a_2 - b_1 b_2})^2 + (a_1 b_2 + a_2 b_1)^2} \\ &= |(\boldsymbol{a_1 a_2 - b_1 b_2}, a_1 b_2 + a_2 b_1)| = |c_1 c_2|. \end{aligned}$$

Reading

- This lecture: chapter 1, p 7-20
- Next lecture (next week?):
  chapter 2   Complex Vector Spaces