April 23, 2007

Challenges in Computer Science Seminar: Intro to Multi-objective Optimization

1 Mathematical Preliminaries

Let p and q be a natural numbers bigger or equal to 1 and let $f_1 : \mathbb{R}^p \to \mathbb{R}$, $f_2 : \mathbb{R}^p \to \mathbb{R}$, ..., $f_q : \mathbb{R}^p \to \mathbb{R}$ be given functions. Let $\mathcal{X} \subseteq \mathbb{R}^p$. We consider problems which fit the following template. Such problems are examples of *multi-objective optimization problems*.

$$\min(f_1(x), \dots, f_q(x)) \text{ subject to } x \in \mathcal{X}. \quad (1)$$

We call $\mathcal{X}$ the feasible set and the image of $\mathcal{X}$ under the objective function mapping $f = (f_1, \dots, f_q)$ is denoted as $\mathcal{Y} := (f_1, \dots, f_q)(\mathcal{X})$.

Note that you can consider max problems as well (and also mixed min and max problems).

Next, we are going to outline notions and definitions of Pareto optimality and non-dominance:

Definition 1.1 *Given two vectors $\mathbf{y} \in \mathbb{R}^q$, $\mathbf{y}' \in \mathbb{R}^q$ we say $\mathbf{y}$ dominates $\mathbf{y}'$ (in symbols: $\mathbf{y} \prec \mathbf{y}'$, iff $\forall i = 1, \dots, q : y_i \leq y'_i$ and $\exists i \in \{1, \dots, q\} : y_i < y'_i$. Moreover, we define $\mathbf{y} \preceq \mathbf{y}' \Leftrightarrow \mathbf{y} \prec \mathbf{y}' \vee \mathbf{y} = \mathbf{y}'$.*

Definition 1.2 *Given a set of points $\mathcal{Y}$, a point $\mathbf{y}$ is said to be non-dominated with respect to $\mathcal{Y}$, iff there does not exist $\mathbf{y}' \in \mathcal{Y} : \mathbf{y}' \prec \mathbf{y}$. Moreover, the subset of non-dominated points $\mathbf{y}$ in $\mathcal{Y}$ with respect to $\mathcal{Y}$ is called the non-dominated set of $\mathcal{Y}$. Also this set is referred to as the Pareto front (PF) of $\mathcal{Y}$.*

In the context of optimization problems $f_i(x) \to \min, i = 1, \dots, q,\ x \in \mathcal{X}$ the concept of dominance is also defined on the search space.

Definition 1.3 *We say $x \prec x' :\Leftrightarrow (f_1(x), \dots, f_q(x)) \prec (f_1(x'), \dots, f_q(x'))$. Also, we define $x \preceq x' :\Leftrightarrow (f_1(x), \dots, f_q(x)) \preceq (f_1(x'), \dots, f_q(x'))$.*

Definition 1.4 *Given a set of points $\mathcal{X}$ the non-dominated subset of the set $\{\mathbf{y} \mid y_1 = f_1(x), \dots, y_q = f_q(x), x \in \mathcal{X}\ \}$ is called the Pareto front (PF) with respect to the optimization problem. Moreover, the inverse image of this set in $\mathcal{X}$ is called the efficient set in $\mathcal{X}$. The elements of this set are called efficient points.*

Example/Problem 1: Consider the problem ($p = 1$ and $q = 2$)

$$\min(f_1(x), f_2(x)) \text{ subject to } x \in \mathbb{R}_+ \text{ positive reals }. \quad (2)$$

, where

$$f_1(x) = \sqrt{1 + x}, f_2(x) = x^2 - 4x + 5. \quad (3)$$

. Determine the image of the feasible set ($\mathcal{Y} = \{(f_1(x), f_2(x)) : x \in \mathcal{X} = \mathbb{R}_+\}$). And determine the non-dominated set $\mathcal{Y}_N := \{y \in \mathcal{Y} :$ there is no $y' \in \mathcal{Y}$ with $y' \prec y\}$ and the efficient set $\mathcal{X}_E := \{x \in \mathcal{X} : f(x) = (f_1(x), f_2(x)) \in \mathcal{Y}_N\}$.

Example/Problem 2: Consider the problem ($p = 1$ and $q = 2$)

$$\min(f_1(x), f_2(x)) \text{ subject to } x \in [-1, 1]. \quad (4)$$

, where

$$f_1(x) = \sqrt{5 - x^2}, f_2(x) = \frac{x}{2}. \tag{5}$$

. Determine the image of the feasible set ($\mathcal{Y} = \{(f_1(x), f_2(x)) : x \in \mathcal{X} = [-1, 1]\}$). And determine the non-dominated set $\mathcal{Y}_N := \{y \in \mathcal{Y} :$ there is no $y' \in \mathcal{Y}$ with $y' \prec y\}$ and the efficient set $\mathcal{X}_E := \{x \in \mathcal{X} : f(x) = (f_1(x), f_2(x)) \in \mathcal{Y}_N\}$.

Intro March 27

Below you will find four optimization problems. The first two are rather academic and serve only to elucidate concepts and transitions. There are several cognitive transitions to consider: 1) from single objective optimization to multi-objective optimization, 2) from a math model can be created and it is feasible to find the optima analytically to a math model can be created but it is not possible to find the information about the optima analytically, and 3) from a math model can be created to no math model is available. Especially the last situation is the reason that we welcome the many new heuristics such as evolutionary approaches, simulated annealing, tabu search etc.

Deliverables

Investigate the third problem very thoroughly. Write a report (using LaTeX) describing your strategy, assumptions, model and findings – include the graphs generated by the tool you have used (i.e, matlab or gnuplot). You can work on this problem in groups of 2 to 3 students. Turn in your report no later than April 23 (e-mail: deutz@liacs.nl); let the subject field contain the string "multi-objective question." Refer questions to me (deutz@liacs.nl).

PROBLEMS

1. (Semi-academic question.) What are the dimensions of an aluminum can that can hold 0.33 liters of juice and that uses the least material (i.e., aluminum)? Assume that the can is cylindrical, and is capped at both ends.
2. (Academic question.) Minimize $f_1(x) = x^2$ and $f_2(x) = (1-x)^2$ "simultaneously" where $x \in \mathbb{R}^+$. What is the shape of the Pareto front?
3. (There is a chance that Michael might approve of this question.) Your task is to design a fish tank. The width, height, and length of the box-shaped tank are your decision variables. The volume has to be maximized, while the surface area has to be minimized because it is directly proportional to the cost of the tank. Moreover, for aesthetical reasons, the ratio between the height and length should be $2/3$. Another constraint: the volume is not to exceed 300 liters and has to be more or equal to 60 liters.
4. Ask me or consult http://www.bionik.tu-berlin.de/institut/xs2arbas/xs2arbas.html

Success