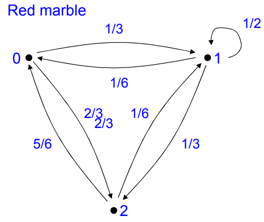
Assembling Systems

Basic Quantum Theory

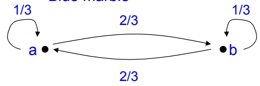
Lecture 4

Assembling Systems

Red marble



Blue marble



$$\text{Adjacency matrix } N = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$\text{Adjacency matrix } M = \begin{bmatrix} 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

$$\text{Typical state } X = \begin{array}{c} 0a \\ 0b \\ 1a \\ 1b \\ 2a \\ 2b \end{array} \begin{bmatrix} \frac{1}{18} \\ 0 \\ \frac{2}{18} \\ \frac{1}{3} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

Dynamics: $i\,j \xrightarrow{M[i',i] \times N[j',j]} i'\,j'$

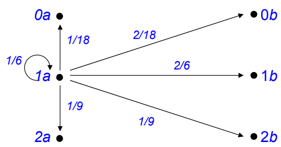
Assembling Systems (cont'd)

$$M \otimes N = \begin{bmatrix} 0\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} & \frac{1}{6}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} & \frac{5}{6}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \\ \frac{1}{3}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} & \frac{1}{2}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} & \frac{1}{6}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \\ \frac{2}{3}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} & \frac{1}{3}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} & 0\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{18} & \frac{2}{18} & \frac{5}{18} & \frac{10}{18} \\ 0 & 0 & \frac{2}{18} & \frac{1}{18} & \frac{10}{18} & \frac{5}{18} \\ \frac{1}{9} & \frac{2}{9} & \frac{1}{6} & \frac{2}{6} & \frac{1}{18} & \frac{2}{18} \\ \frac{2}{9} & \frac{1}{9} & \frac{2}{6} & \frac{1}{6} & \frac{2}{18} & \frac{1}{18} \\ \frac{2}{9} & \frac{4}{9} & \frac{1}{9} & \frac{2}{9} & 0 & 0 \\ \frac{4}{9} & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & 0 & 0 \end{bmatrix}$$

Corresponding graph

Cartesian product G_M x G_N

(only third column of tensor product of M and N)



Assembling Systems (cont'd)

- Quantum theory:
  - States can be combined using the tensor product of the vectors
  - Changes of the system are combined by using the tensor product of matrices
  - Important: there are many more states that cannot be combined from "smaller" ones:
    - No tensor product of smaller states
    - More interesting ones
    - Called entangled states
  - Similar for actions

Assembling Systems (cont'd)

- In general:
  - Cartesian product of an n-vertex graph with an n'-vertex graph is an (n x n')-vertex graph.
  - If we have an n-vertex graph and we are interested in m different marbles within this system, this results in the graph with n^m vertices

$$G^m = \underbrace{G \times G \times \cdots \times G}_{m \text{ times}}$$

  - with the associated n^m-by-n^m adjacency matrix

$$M_G^{\otimes m} = \underbrace{M_G \otimes M_G \otimes \cdots \otimes M_G}_{m \text{ times}}$$

- Example: bit as a two-vertex graph with a marble on the 0 vertex or a marble on the 1 vertex. For m bits with a single marble one needs a 2^m vertex graph or a 2^m-by-2^m matrix, which demonstrates an exponential growth.

Basic Quantum Theory

Why Quantum Mechanics?

- Classical mechanics:
  - Dichotomy: particles (matter) ↔ waves (light)
  - Several experiments prove falseness
- New theory of microscopic world: *both matter and light manifest a particle-like and a wave-like behavior.*
- Double-slit experiment:
  - Also with just one photon: which region is more likely for the single photon to land. The photon is a true chameleon: sometimes it behaves as a particle and sometimes as a wave, depending on how it is observed.
  - Not only for light (photons), but also with electrons, protons, and even atomic nuclei. Clearly indicates: no rigid distinction between waves and particles.

Quantum states

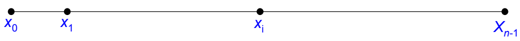
Two examples:

I. A particle confined to a set of discrete positions on a line

II. A single-particles spin system

Classical state

Subatomic particle on a line: can only be detected at one of the equally spaced points $\{x_0, x_1, \ldots, x_{n-1}\}$, where $x_1 = x_0 + \delta x$, $x_2 = x_1 + \delta x$.



δx can be made as small as one wishes.

Associate to this current state of the particle an n-dimensional complex column vector $[c_0, c_1, \ldots, c_{n-1}]^T$. Particle at point x_i shall be denoted by the Dirac ket notation $|x_i>$.

To each of these n basic states, we assiciate:

$$| x_0\rangle \mapsto [1, 0, \ldots, 0]^T$$
$$| x_1\rangle \mapsto [0, 1, \ldots, 0]^T$$
$$\vdots$$
$$| x_{n-1}\rangle \mapsto [0, 1, \ldots, 0]^T$$

Canonical basis of C^n

Classical viewpoint: that's all we need!

Quantum state

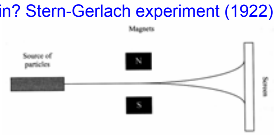
- Experiments demonstrate that the particle can be in a strange fuzzy blending of these states. What does this mean?
- An arbitrary state (c_i's are the complex amplitudes)
  $$|\phi\rangle = c_0 \,| x_0\rangle + c_1 \,| x_1\rangle + \cdots + c_{n-1} \,| x_{n-1}\rangle$$
- Represented in $\mathbf{C}^n$ as $|\phi\rangle \mapsto [c_0, c_1, \ldots, c_{n-1}]^T$
- This state is a superposition of the basis states: it is simultaneously in all $\{x_0, x_1, \ldots, x_{n-1}\}$ locations.
- The complex numbers $c_0, c_1, \ldots, c_{n-1}$ tells precisely which superposition our particle is in.
- We will detect the particle in point x_i with a probability $p(x_i) = \frac{|c_i|^2}{\| \phi\rangle|^2} = \frac{|c_i|^2}{\sum_j |c_j|^2}$
- After the observation we will find the system in one of the basis states
  $$|\phi\rangle \rightsquigarrow | x_i\rangle$$

Properties of kets

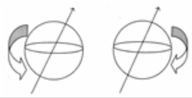
- Kets can be added
  $$|\phi\rangle + |\phi'\rangle = (c_0 + c'_0)\,| x_0\rangle + \cdots + (c_{n-1} + c'_{n-1})\,| x_{n-1}\rangle$$
- Scalar multiply a ket by c
  $$c\,|\phi\rangle = cc_0\,| x_0\rangle + \cdots + cc_{n-1}\,| x_{n-1}\rangle$$
- A ket and all its complex scalar multiples describe the same physical state. So the length of a ket does not matter as far as physics goes.
- A normalized ket $\frac{|\phi\rangle}{\| \phi\rangle|}$
- Given a normalized ket, we get $p(x_i) = |c_i|^2$

Spin

- What is spin? Stern-Gerlach experiment (1922)



- Split of beam electrons into two streams with opposite spin. Wrt a classical spinning top two striking differences:
  - Electron does not have an internal structure, it is just a charged point.
  - All electrons either at the top or at the bottom. Two states: it spins either clockwise or anticlockwise.
- Two basic spin states: spin **up** and **down**
  $$|\phi\rangle = c_0\,|\uparrow\rangle + c_1\,|\downarrow\rangle$$



Bra-ket

- Physical meaning of inner product: transition amplitudes, how likely will the state of the system change *before* a specific measurement (start state) to another (end state) *after* the measurement.

- How to calculate a transition amplitude?

$$\text{start state } |\phi\rangle = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} \quad \text{and end state } |\phi'\rangle = \begin{bmatrix} c_0^{'} \\ c_1^{'} \\ \vdots \\ c_{n-1}^{'} \end{bmatrix}$$

  - Bra state: $\langle \phi^{'} | = |\phi^{'}\rangle^{\dagger} = [\overline{c_0^{'}}, \overline{c_1^{'}}, \ldots, \overline{c_{n-1}^{'}}]$
  - Transition amplitude: multiply as *matrices*

$$\langle \phi^{'} | \phi \rangle = [\overline{c_0^{'}}, \overline{c_1^{'}}, \ldots, \overline{c_{n-1}^{'}}] \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} = \overline{c_0^{'}} \times c_0 + \overline{c_1^{'}} \times c_1 + \cdots + \overline{c_{n-1}^{'}} \times c_{n-1}$$

  - Denoted as: $|\phi\rangle \xrightarrow{\langle \phi^{'} | \phi \rangle} |\phi^{'}\rangle$
  - Nothing else than the inner product: from states to state transitions.

Summary Quantum States

- Association of a vector space to a quantum space. The dimension reflects the amount of basis states of the system.

- States can be superposed, by adding their representing vectors.

- A state is left unchanged if its representing vector is multiplied by a complex scalar.

- The state space has a geometry, given by its inner product. This geometry has a physical meaning: it tells us the likelihood for a given state to transition into another one after being measured. States that are orthogonal to one another are mutually exclusive.

Observables

- To each physical observable there corresponds a hermitian operator.

  - An observable is a linear operator, which means it maps states to states. Apply Ω to the state vector |ψ>, the resulting state is Ω |ψ>.

  - The eigenvalues of a hermitian operator are all real.

- The eigenvalues of a hermitian operator Ω associated with a physical observable are the only possible values the observable can take as a result of measuring it on a given state. Furthermore, the eigenvectors of Ω form a basis for the state space.

Position observable

- Where can the particle be found?

- Acts on the basic states:

  - $P(|x_i\rangle) = x_i\, |x_i\rangle$ *P* acts as multiplication by position.

- On arbitrary states: $P|\phi\rangle = P\left(\sum c_i\, |x_i\rangle\right) = \sum x_i c_i\, |x_i\rangle$

- Matrix representation of the operator in the standard basis:

$$P = \begin{bmatrix} x_0 & 0 & \cdots & 0 \\ 0 & x_1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{n-1} \end{bmatrix}$$

Momentum observable

- Classical: momentum = velocity x mass

- Quantum analog: $M(|\phi\rangle) = -i * \hbar * \dfrac{|\phi(x+\delta x)\rangle - |\phi(x)\rangle}{\delta x}$

  - Is the rate of change of the state vector from one point to the next.
  - The constant ħ (pronounced h bar) is a universal constant, called the reduced Planc constant.

- Many more observables, but position and momentum are in a sense building blocks.

Spin operators

- Given a direction in space, in which way is the particle spinning?

- Is the particle spinning up or down in the *z* direction? Left or right in the *x* direction? In or out in the *y* direction?

- The three corresponding operators:

$$S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Orthonormal bases:
  - S_z has eigenbasis up and down $\{|\uparrow\rangle, |\downarrow\rangle\}$
  - S_y has eigenbasis in and out $\{|\swarrow\rangle, |\nearrow\rangle\}$
  - S_x has eigenbasis left and right $\{|\leftarrow\rangle, |\rightarrow\rangle\}$

More on operators/observables

- p117-125: FYI, not really important for quantum computation

Sum up on observables

- Observables are represented by hermitian operators. The result of an observation is always an eigenvalue of the hermitian.
- The expression $<\psi|\Omega|\psi>$ represents the expected value of observing Ω on $|\psi>$.
- Observables in general do not commute. This means that the order of observations matters. Moreover, if the commutator of two observables is not zero, there is an intrinsic limit to our capability of measuring their values simultaneously.

Measuring

- The act of carrying out an observation on a given physical system is called measuring.
- Classical:
  - Measuring leaves the system in whatever state it already was, at least in principle.
  - The result of a measurement on a well-defined state is predictable.
- Quantum world:
  - Systems do get perturbed and modified as a result of a measurement.
  - Only the probability of observing specific values can be calculated: measurement is inherently a nondeterministic process.

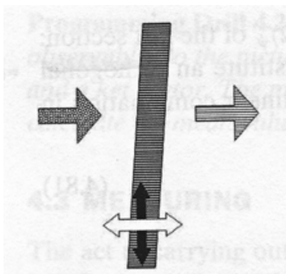
What happens?

- Let Ω be an observable and $|\psi>$ be a state. If the result of measuring Ω is the eigenvalue λ, the state after measurement will always be an eigenvector corresponding to λ.
- The probability of the transition to an eigenvector is equal to $|<e|\psi>|^2$. It is the projection of $|\psi>$ along $|e>$.

Measurement with more than one observable

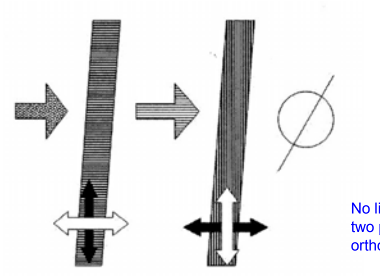
- Beam of light:
  - Vibrates along all possible directions orthogonal to its line of propagation.
  - Vibrates only in a specific direction: polarization.
- Experiment: multiple polarization sheets.

One sheet



Light partially passing through one polarization sheet.

Two sheets

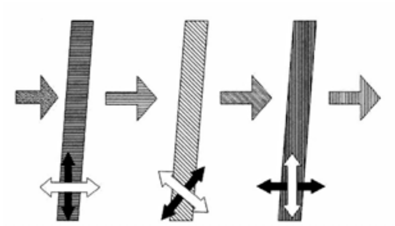


No light passing through two polarization sheets at orthogonal angles.

Three sheets

No effect if third sheets is placed on left or right of the two other sheets: no light!

However, placed in-between: only one-eight of the original light will pass through all three sheets!



Summary on measuring

- The end state of the measurement of an observable is always one of its eigenvectors.
- The probability for an initial state to collapse into an eigenvector of the observable is given by the length squared of the projection.
- When we measure several observables, the order of measurement matters.

Quantum dynamics

- Systems evolving in time.
- The evolution of a quantum system (that is not a measurement) is given by a unitary operator or transformation

$$|\phi(t+1)\rangle = U\,|\phi(t)\rangle$$

- Unitary transformations are closed under composition and inverse:
  - The product of two arbitrary unitary matrices is unitary.
  - The inverse of a unitary transformation is unitary.

Quantum dynamics (cont'd)

- Assume we have a rule $\Re$ that associates with each instance of time $t_0, t_1, t_{2,} \ldots, t_{n-1}$

  a unitary matrix $\Re[t_0], \Re[t_1], \ldots, \Re[t_{n-1}]$

- Starting with an initial state vector $|\phi\rangle$

$$\begin{aligned}
&\Re[t_0]\,|\phi\rangle,\\
&\Re[t_1]\Re[t_0]\,|\phi\rangle,\\
&\quad\vdots\\
&\Re[t_{n-1}]\Re[t_{n-2}]\cdots\Re[t_0]\,|\phi\rangle
\end{aligned}$$

Quantum dynamics (cont'd)

Orbit of $|\psi\rangle$

$$|\phi\rangle \underset{\Re[t_0]^\dagger}{\overset{\Re[t_0]}{\rightleftarrows}} \Re[t_0]\,|\phi\rangle \underset{\Re[t_1]^\dagger}{\overset{\Re[t_1]}{\rightleftarrows}} \Re[t_1]\Re[t_0]\,|\phi\rangle \underset{\Re[t_2]^\dagger}{\overset{\Re[t_2]}{\rightleftarrows}} \Re[t_2]\Re[t_1]\Re[t_0]\,|\phi\rangle$$

$$\rightleftarrows \cdots \underset{\Re[t_{n-1}]^\dagger}{\overset{\Re[t_{n-1}]}{\rightleftarrows}} \Re[t_{n-1}]\Re[t_{n-2}]\cdots\Re[t_0]\,|\phi\rangle$$

Symmetric in time

A quantum computation will start with an initial state $|\psi\rangle$, followed by the application of a sequence of unitary operators to that state. When we are done, we will measure the output and get the final state.

Quantum dynamics (cont'd)

- How is the sequence of unitary transformations selected in real-life quantum mechanics?
- How is the dynamics determined?
- How does the system change?

- Answer: the Schrödinger equation (see book)

Quantum dynamics: recap

- Quantum dynamics is given by unitary transformations.

- Unitary transformations are invertible: thus, all closed system dynamics are reversible in time (as long as no measurement is involved).

- The concrete dynamics is given by the Schrödinger equation, which determines the evolution of a quantum system.

Reading

- This lecture: Ch 3.4 & Ch 4.1-4.4, p 97-132.

- Next lecture: Ch 4.5 & start Ch 5.